Degrees of Deducibility

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CONTENTS

Summary (to be continued)

Not all inferences are deductively valid, but some seem to be closer to validity than others are. It is not so much a commonplace as an unstudied presumption that the appropriate way to grade the validity of an inference from a premise or assumption a to a conclusion c is by means of a logical (epistemic, judgemental) probability measure: the probability $\mathfrak{p}(c \mid a)$ takes the value 1 when the inference is classically valid, and generally a lower non-negative value when it is invalid.

It will be argued here that this presumption is contentious, and worth calling into question, since there exist several other functions, definable in terms of the values that a probability measure $\mathfrak{p}$ takes on pairs of truth functions $\langle Z, X \rangle$ of a and c, that provide tenable necessary conditions for the deducibility of c from a; and, when the measure $\mathfrak{p}$ is strictly positive, necessary & sufficient conditions for deducibility.

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Summary (concluded)

This lecture aims to impugn the privileged position of the probability function $\mathfrak{p}$ by determining which other functions $\mathfrak{f}(c \mid a) = \mathfrak{p}(Z \mid X)$, where X and Z are truth functions of a and c, take the value $\mathbf{1}$ when c is deducible from a (and conversely when $\mathfrak{p}$ is regular). The 27 candidates paraded initially will rapidly be reduced to eight essentially different functions, two of which will be disqualified on formal grounds.

Three of the remaining six functions will recommend themselves as equally deserving of attention, and in a mysterious way interchangeable. One of the three is the probability function $\mathfrak{p}$. A second, which I now call the contraprobability function $\mathfrak{q}(c \mid a) = \mathfrak{p}(a' \mid c')$, has been promoted elsewhere in connection with both confirmation and conditionals. The third function $\mathfrak{n}(c \mid a) = \mathfrak{p}(a' \mid a \triangle c)$, which is formally a (directed) proximity function, provokes many teasing questions.

1 A fateful step

In legal circles objective comparisons of degrees of validity or of proof have long been thought to be desirable, though perhaps not quantitatively. According to Hacking (1975), it may have been the young Leibniz in the 1660s who was the first to invest in the rapidly ripening mathematical calculus of chance — what we now call the theory of probability — in the expectation that it would effortlessly yield a definitive calculus of opinion.

The impulsive adoption of this mathematical formalism caught on. It is the moral, if not the thesis, of this lecture that neither logic nor the law has recovered from it.

1 Logical probability

At least since Bolzano's Wissenschaftslehre (1837), it has been habitual to dream of evaluating the degrees of validity of an inference that is not deductively valid in terms of how probable its conclusion is made by its premises.

The Spielraum or range theory of von Kries (1886) explicitly supposes that in addition to the physical probabilities that are realized in statistical frequencies in nature, there is an equally objective kind of probability that is more logical in character, and, at least in some cases, determinate in value a priori. In one version or another, this doctrine of logical probability has continued to be popular.

1 The logical interpretation of probability

Kries's theory was criticized by Keynes (1921), but soon revived by Waismann (1930), under the influence of Wittgenstein, and adopted with reservations also by Popper in Logik der Forschung (1935), by Kneale (1947), and by Carnap in Logical Foundations of Probability (1950).

The guiding idea of this logical interpretation of the calculus of probability is that $\mathfrak{p}(c \,|\, a) = 1$ states a necessary connexion between the sentences a and c, that is, the deducibility of the conclusion c from the assumption or premise a. Lower values of $\mathfrak{p}$ indicate that the connexion between a and c falls in some respect short of necessity.

1 Probability measures the range of possibilities

The **Spielraum** theory understands the function $\mathfrak{p}$ to be an abstract measure on **ranges** or **scopes**, that is, the sets of possibilities (possible worlds) that can be distinguished in the language under consideration (or the maximal consistent sets of sentences in that language). The relative (or conditional) probability $\mathfrak{p}(c \mid a)$ is identified with the proportion (in a suitably generalized sense) of those possibilities admitted by a that are admitted also by c.

It is evident that $\mathfrak{p}$ achieves its maximum 1 when all possibilities admitted by a are admitted by c, and it is intended also that it should achieve its minimum 0 when none is.

1 Waismann

In other words, $\mathfrak{p}(c\,|\,a) = 1$ when the range of c includes the range of a, and $\mathfrak{p}(c\,|\,a) = 0$ when the range of c excludes the range of a. In short, $\mathfrak{p}(c\,|\,a) = 1$ when c is deducible from a, and $\mathfrak{p}(c\,|\,a) = 0$ when the negation of c, here written c', is deducible from a. 'Entailment and contradiction are represented in this picture as . . . topological relations between scopes' (Waismann).

But this will not do in the event that the range of the sentence $a = \varnothing$, that is, a is contradictory, or is interdeducible with the falsum $\bot$. For then both c and c' are deducible from a, and hence $0 = 1$, which is intolerable.

1 Failure of the usual law of complementation

Most writers therefore disallow contradictory second arguments in $\mathfrak{p}$ from the outset, a strategy that seriously impoverishes the pretensions of the theory of logical probability to generalize the relation of classical deducibility.

A more polished solution (Popper 1959) is to allow the deducibility of c from $\bot$ to override its exclusion by $\bot$, that is, to set $\mathfrak{p}(c \mid \bot) = 1$ for all conclusions c. The familiar law of complementation $\mathfrak{p}(c \mid a) + \mathfrak{p}(c' \mid a) = 1$ holds only for consistent assumptions a. But $\mathfrak{p}(c \mid a) = 1$ remains valid whenever c is deducible from a (in particular $\mathfrak{p}(b \mid b) = 1$), and accordingly $\mathfrak{p}(c \mid \bot) + \mathfrak{p}(c' \mid \bot) = 2$.

2 Elementary theory of probability

We shall adopt as a set of axioms for the probability function $\mathfrak{p}$ the system given by Popper in The Logic of Scientific Discovery, appendix ∗v. The range of $\mathfrak{p}$ is assumed to be the field of real numbers, and adequate axioms for real arithmetic are assumed implicitly.

The system is more general than the standard system of Kolmogorov, in which the singular operator $\mathfrak{p}(b)$ is primitive, and $\mathfrak{p}(c \mid a) =_{\text{df}} \mathfrak{p}(ca)/\mathfrak{p}(a)$ only when $\mathfrak{p}(a) > 0$. It is more general even than the systems of Hosiasson-Lindenbaum and of Rényi, in which the binary operator $\mathfrak{p}(c \mid a)$ is primitive. Here $\mathfrak{p}(c \mid a)$ is defined for every a, c.

2 Popper's axioms

A0	$\exists c \exists d\, \mathfrak{p}(a \mid c) \neq \mathfrak{p}(b \mid d)$
A1	$\forall b(\mathfrak{p}(a \mid b) = \mathfrak{p}(c \mid b)) \Rightarrow \mathfrak{p}(b \mid a) = \mathfrak{p}(b \mid c)$
A2	$\mathfrak{p}(a \mid a) = \mathfrak{p}(c \mid c)$
B1	$\mathfrak{p}(ac \mid b) \leq \mathfrak{p}(a \mid b)$
B2	$\mathfrak{p}(ac \mid b) = \mathfrak{p}(a \mid cb)\mathfrak{p}(c \mid b)$
C	$\mathfrak{p}(a \mid a) \neq \mathfrak{p}(b \mid a) \Rightarrow \mathfrak{p}(a \mid a) = \mathfrak{p}(c \mid a) + \mathfrak{p}(c' \mid a).$

Given $a \vee c =_{\text{df}} (a'c')'$, **we may prove also the general addition law** $\mathfrak{p}(ac \mid b) + \mathfrak{p}(a \vee c \mid b) = \mathfrak{p}(a \mid b) + \mathfrak{p}(c \mid b)$.

2 Probabilistic indistinguishability

There are no algebraic axioms constraining the two operators (concatenation and $'$). The system is **autonomous**. Yet the domain reduces to a Boolean algebra when factored by the congruence $a \sim c =_{df} \forall b(\mathfrak{p}(a|b) = \mathfrak{p}(c|b))$.

We shall work in the other direction, and assume that the relations $\vdash$ and $\equiv$ of deducibility and interdeducibility are given. We shall admit only measures $\mathfrak{p}$ under which **logically distinguishable sentences are those distinguishable in measure**; that is, that a and c are interdeducible if & only if $\mathfrak{p}(a \mid b) = \mathfrak{p}(c \mid b)$ for every sentence b. $\forall b\, \mathfrak{p}(c|b) \geq \mathfrak{p}(a|b)$ is necessary & sufficient for $a \vdash c$.

2 Strictly positive measures

That for every denumerable language there is such a measure $\mathfrak{p}$ follows from the existence of a **regular** or **strictly positive** measure. It is easy to see that if $\mathfrak{p}$ is regular then logically distinguishable sentences are exactly those distinguishable in measure. It is less easy to see that the converse fails. Only in some examples shall we assume that $\mathfrak{p}$ is regular. In general, $\mathfrak{p}(c \,|\, a) = 1$ will be necessary but insufficient for the deducibility of c from a.

There is still a clear sense in which $\mathfrak{p}(c \,|\, a)$ is a measure of the **degree of deducibility** of c from a, equalling 1 when c is deducible from a, and 0 when c' is deducible from a.

3 Credence distinguished from probability

We turn now to our central question: are there other functions $\mathfrak{f}(c \mid a)$, definable in terms of the probability function $\mathfrak{p}$, that can serve in a natural way as measures of the degree to which a sentence c is deducible from a sentence a? If so, how many such functions are there?

For clarity we shall henceforth use the term probability $\mathfrak{p}$ only to refer to abstract measures obeying the axioms given above. We shall refer to the customary epistemic or subjectivist or judgemental (Jeffrey 1986) interpretation of $\mathfrak{p}$ as some kind of measure of degree of belief as credence $\mathfrak{c}$, despite the unwelcome psychological flavour.

3 A disclaimer

The objective of this lecture is to call into doubt the thesis that the Spielraum theory deserves to be regarded as the unique way of measuring degrees of (classical) deducibility. I shall do this by showing that some heretofore neglected measures may do the job as well, or even better.

It is incontestable that abstract probability measures can be imposed on the sentences of many formalized languages. But the assumption that these measures, or functions defined in their terms, can be profitably interpreted as credences, or in any other way, is an assumption made by my opponents, and not one for which I am answerable.

3 Another function definable from $\mathfrak{p}$

Since the two functions $\mathfrak{p}$ and $\mathfrak{c}$ are numerically identical (in the arithmetical, not the metaphysical, sense), $\mathfrak{c}(c \mid a)$ is one way to construe the degree to which the conclusion c of an inference is deducible from the assumption a.

Elsewhere I have identified another generalization of the relation of classical deducibility, the measure $\mathfrak{q}$ of deductive dependence or contraprobability. The function $\mathfrak{q}$ too is definable from the measure $\mathfrak{p}$, by the formula $\mathfrak{q}(a \mid c) = \mathfrak{p}(c' \mid a')$. It is not hard to show that $\mathfrak{c}$ and $\mathfrak{q}$ generally take different values, except of course when c is deducible from a, in which case each takes the value 1.

3 Statements equivalent to $\mathfrak{a} \vdash \mathfrak{c}$

The stimulus for interpreting the function $\mathfrak{q}$ as a measure of deducibility was the logical truism that $c' \vdash a'$ is as good a way of stating the deducibility of c from a as $a \vdash c$ is. In addition to grading the deducibility of c from a in terms of the proportion of possibilities admitted by a that are admitted by c, as $\mathfrak{p}(c \,|\, a)$ does, we can therefore grade it by the proportion of possibilities admitted by c' that are admitted also by a'. This is what $\mathfrak{q}(c \,|\, a)$ does.

Our question may be instructively sharpened: are there other equivalent statements of the deducibility of c from a that give rise interesting measures definable in terms of $\mathfrak{p}$?

3 Two disappointing examples

By the deduction theorem $\mathfrak{a} \vdash \mathfrak{c}$ if & only if $\vdash \mathfrak{a} \to \mathfrak{c}$; that is, if & only if $\top \vdash \mathfrak{a} \to \mathfrak{c}$. We can therefore grade the deducibility of $\mathfrak{c}$ from $\mathfrak{a}$ by the proportion of possibilities admitted by $\top$ that are admitted by $\mathfrak{a} \to \mathfrak{c}$. Using $\mathfrak{t}$ for tautology, we write $\mathfrak{t}(\mathfrak{c}\,|\,\mathfrak{a})$ for $\mathfrak{p}(\mathfrak{a} \to \mathfrak{c}\,|\,\top)$. It will later be shown that $\mathfrak{t}$ is not a genuine generalization of deducibility.

By reductio ad absurdum, $\vdash \mathfrak{a} \to \mathfrak{c}$ if & only if $\mathfrak{a}\mathfrak{c}' \vdash \bot$. We can therefore grade the deducibility of $\mathfrak{c}$ from $\mathfrak{a}$ by the function $\mathfrak{r}(\mathfrak{c}\,|\,\mathfrak{a}) = \mathfrak{p}(\bot\,|\,\mathfrak{a}\mathfrak{c}')$, whence $\mathfrak{r}(\mathfrak{c}\,|\,\mathfrak{a})$ equals 1 when $\mathfrak{a} \vdash \mathfrak{c}$, and equals 0 when $\mathfrak{a} \nvdash \mathfrak{c}$. This shows that the function $\mathfrak{r}$ is not a generalization of deducibility at all.

3 The initiative of Jonathan Cohen

To my knowledge, the first author to suggest explicitly that '**if provability is to be put on a scale and made a matter of degree, there are at least two prima-facie candidates for its lower limit**' was Jonathan Cohen (1977), who continued: '[o]ne is disprovability, the other non-provability'.

Orthodox probability (which Cohen calls '**Pascalian probability**') indeed has disprovability as its lower limit (if by **prove** is meant **deduce**), but non-provability is the lower limit of the **reductio function** $\mathfrak{r}$ just described, rather than Cohen's innovative '**non-Pascalian probability**'. I intend on another occasion to evaluate Cohen's ideas more sympathetically.

3 Necessary & sufficient conditions for deducibility

In the following systematic investigation of functions that are defined from probability $\mathfrak{p}$ and, like $\mathfrak{c}(c\,|\,a)$, $\mathfrak{q}(c\,|\,a)$, and $\mathfrak{t}(c\,|\,a)$, can be said in some broad sense to measure the degree to which c is deducible from a, it must be remembered that $\mathfrak{c}(c\,|\,a) = 1$, $\mathfrak{q}(c\,|\,a) = 1$, and $\mathfrak{t}(c\,|\,a) = 1$ are only necessary conditions for c to be deducible from a, though necessary & sufficient if $\mathfrak{p}$ is a regular measure.

For each function we need to show also how simple deducibility is definable in its terms. In the case of the credence function $\mathfrak{c}$ Popper showed that $a \vdash c$ is defined by $\forall b\ \mathfrak{c}(a\,|\,b) \leq \mathfrak{c}(c\,|\,b)$ (or $\forall b\ \mathfrak{c}(c\,|\,ab) = 1$, or $\mathfrak{c}(c\,|\,ac') = 1$).

4 A search for truth functions

We have noted that $a \vdash c$, and $c' \vdash a'$, and $\top \vdash a \to c$, and $ac' \vdash \bot$ are equivalent ways of saying that c is deducible from a. Our first task is to complete this list by finding all those **truth functions** (or **Boolean polynomials**) X and Z in a and c for which $X \vdash Z$ **if & only if** $a \vdash c$. Logically equivalent truth functions need not be distinguished.

It turns out to be somewhat easier to replace talk in terms of deducibility by talk in terms of the corresponding material conditionals. Our task is then to identify those X, Z for which the conditional $X \to Z$, viewed as a truth function of a and c, has the same truth table as the conditional $a \to c$.

4 The possible tables for X and Z

Provided that we identify truth functions that are logically equivalent, there are 27 distinct pairs of truth functions X and Z in the letters a, c such that $X \rightarrow Z$ and $a \rightarrow c$ are logically equivalent; that is, have identical truth tables.

a	c										
T	T		T	T	T	T	F	F	F	F	
T	F		T	T	T	T	T	T	T	T	
F	T		T	T	F	F	T	T	F	F	X
F	F		T	F	T	F	T	F	T	F	
T	T		T	T	T	T					
T	F		F	F	F	F	F	F	F	F	Z
F	T		T	T			T	T			
F	F		T		T		T		T		

4 Explanation of the tables for X and for Z

Since $a \rightarrow c$ has the value F if & only if a has the value T and c has the value F, the same holds for $X \rightarrow Z$. It follows that the second row of X's truth table cannot be F, for then $X \rightarrow Z$ would be T; and if its first (or third, or fourth) row is T, then the first (or third, or fourth) row of Z's table cannot be F, for then $X \rightarrow Z$ would be F.

There are eight truth functions X of a and c whose second row is not F. They are shown in the upper section of the table, In the lower section are shown the possible distributions of truth values for the truth function Z. Where there is a blank space, each of T and F is a solution.

4 The 27 pairs of truth functions X and Z

Each antecedent X is linked with one, two, four, or eight consequents Z. The symbols $\uparrow$, $\downarrow$, $\triangle$ stand for Sheffer stroke (nand), joint denial (nor), and symmetric difference (niff).

X	$\top$	$a \vee c$	$c \to a$	a	$a \uparrow c$	$a \triangle c$	c'	ac'
$\to$	$a \to c$	$a \to c$	$a \to c$	$a \to c$	$a \to c$	$a \to c$	$a \to c$	$a \to c$
		c	$a \leftrightarrow c$	c	a'	c	$a \leftrightarrow c$	c
				$a \leftrightarrow c$		a'	a'	$a \leftrightarrow c$
				ac		$a'c$	$a \downarrow c$	ac
Z								a'
								$a'c$
								$a \uparrow c$
								$\bot$

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4 Dropping 19 pairs

It is because we want to consider $\mathfrak{p}(Z|X)$ as a measure of the degree to which c is deducible from a that we are interested in truth functions X and Z for which $X \to Z$ and $a \to c$ are logically equivalent. In this context, the 27 pairs of truth functions exhibited in the table above may be reduced to eight, one for each of the eight antecedents $\top$, $a \vee c$, $c \to a$, a, $a \uparrow c$, $a \triangle c$, c', ac' in the top row.

This is because if $X \to Y$ and $X \to W$ are logically equivalent (in particular, equivalent to $a \to c$), the identities $\mathfrak{p}(Y|X) = \mathfrak{p}(XY|X) = \mathfrak{p}(X(X \to Y)\,|X) = \mathfrak{p}(X(X \to W)\,|X) = \mathfrak{p}(XW|X) = \mathfrak{p}(W|X)$ hold, whence $\mathfrak{p}(Y|X) = \mathfrak{p}(W|X)$.

4 Eight measures of deducibility

We are left with eight essentially distinct pairs X, Z for which $\mathsf{X} \to \mathsf{Z}$ and $\mathfrak{a} \to \mathfrak{c}$ are logically equivalent, and therefore with eight possible measures of degree of deducibility.

measure	in terms of $\mathfrak{p}$	mnemonic
$\mathfrak{t}(c \mid a)$	$\mathfrak{p}(a \to c \mid \top)$	tautology
$\mathfrak{a}(c \mid a)$	$\mathfrak{p}(c \mid a \vee c)$	alternation
$\mathfrak{b}(c \mid a)$	$\mathfrak{p}(a \to c \mid c \to a)$	biconditional
$\mathfrak{c}(c \mid a)$	$\mathfrak{p}(c \mid a)$	credence
$\mathfrak{s}(c \mid a)$	$\mathfrak{p}(a' \mid a \uparrow c)$	Sheffer stroke
$\mathfrak{n}(c \mid a)$	$\mathfrak{p}(a' \mid a \vartriangle c)$	nearness
$\mathfrak{q}(c \mid a)$	$\mathfrak{p}(a' \mid c')$	deductive dependence
$\mathfrak{r}(c \mid a)$	$\mathfrak{p}(\bot \mid ac')$	reductio

4 Lower limits of $\mathfrak{c}$ and $\mathfrak{q}$

By construction each of these eight measures takes the value 1 when c is deducible from a. This condition is also a necessary one when the probability measure $\mathfrak{p}$ is regular.

As for the lower limit, namely 0, we know that the credence $\mathfrak{c}(c\,|\,a)$ takes the value 0 provided that both $a \vdash c'$ and $a \nvdash c$. If $\mathfrak{p}$ is a regular measure, these conditions are both necessary for $\mathfrak{c}(c\,|\,a) = 0$. We know also that the deductive dependence $\mathfrak{q}(c\,|\,a)$ takes the value 0 provided that both $a' \vdash c$ (which is another way of saying that a and c are subcontraries) and $a \nvdash c$. Again, these conditions are both necessary for $\mathfrak{q}(c\,|\,a) = 0$ if $\mathfrak{p}$ is regular.

4 Lower limits of all eight measures

In the table below the rows are been reordered. When $\mathfrak{p}$ is a regular measure, the conditions given in the final column are, as before, necessary for the value $\mathbf{0}$ to be assumed.

measure	in terms of $\mathfrak{p}$	when equal to $\mathbf{0}$
$\mathfrak{t}(c\,\vert\,a)$	$\mathfrak{p}(a \to c\,\vert\,\top)$	$a \nvdash c$ and $a \equiv c' \equiv \top$
$\mathfrak{b}(c\,\vert\,a)$	$\mathfrak{p}(a \to c\,\vert\,c \to a)$	$a \nvdash c$ and $a \equiv c'$
$\mathfrak{a}(c\,\vert\,a)$	$\mathfrak{p}(c\,\vert\,a \vee c)$	$a \nvdash c$ and $c' \equiv \top$
$\mathfrak{s}(c\,\vert\,a)$	$\mathfrak{p}(a'\,\vert\,a \uparrow c)$	$a \nvdash c$ and $a \equiv \top$
$\mathfrak{c}(c\,\vert\,a)$	$\mathfrak{p}(c\,\vert\,a)$	$a \nvdash c$ and $a \vdash c'$
$\mathfrak{q}(c\,\vert\,a)$	$\mathfrak{p}(a'\,\vert\,c')$	$a \nvdash c$ and $a' \vdash c$
$\mathfrak{n}(c\,\vert\,a)$	$\mathfrak{p}(a'\,\vert\,a \bigtriangleup c)$	$a \nvdash c$ and $c \vdash a$
$\mathfrak{r}(c\,\vert\,a)$	$\mathfrak{p}(\bot\,\vert\,ac')$	$a \nvdash c$

4 The reordered rows

It is easily checked that when $\mathfrak{t}$ takes the value 0 then so do the seven measures below it; that when $\mathfrak{b}$ takes the value 0 then so do $\mathfrak{c}$ and $\mathfrak{q}$; that when $\mathfrak{a}$ takes the value 0 then so do $\mathfrak{c}$ and $\mathfrak{n}$; that when $\mathfrak{s}$ takes the value 0 then so do $\mathfrak{q}$ and $\mathfrak{n}$; and, trivially, that when any of the seven measures above it takes the value 0, then $\mathfrak{r}$ does so too. But only when $a \equiv \top$ and $c \equiv \bot$ can the two elements in any of the pairs $\{\mathfrak{b}, \mathfrak{n}\}$, $\{\mathfrak{a}, \mathfrak{q}\}$, and $\{\mathfrak{s}, \mathfrak{c}\}$ each equal 0.

We shall soon see that this partial ordering of the eight potential measures of degree of deducibility is preserved intact under a comparison of their numerical magnitudes.

4 A tripod of opposition

Setting to one side the singularities that arise when a is contradictory, c is tautological, or a and c are logically equivalent, the functions $\mathfrak{c}(\mathrm{c} \mid \mathrm{a})$, $\mathfrak{q}(\mathrm{c} \mid \mathrm{a})$, and $\mathfrak{n}(\mathrm{c} \mid \mathrm{a})$ together assume the value $\mathbf{1}$ when $\mathrm{a} \vdash \mathrm{c}$, and separately assume the value $\mathbf{0}$ when $\mathrm{a} \vdash \mathrm{c}'$, $\mathrm{a}' \vdash \mathrm{c}$, and $\mathrm{c} \vdash \mathrm{a}$, respectively.

Here are represented the four simplest deductive relationships that can hold between sentences c and a. With a nod at traditional terminology (and also at Sheffer 1926) we may label them superalternation (or implication) $\mathrm{c} \vdash \mathrm{a}$, contrariety (or incompatibility) $\mathrm{a} \vdash \mathrm{c}'$, subcontrariety (or independence) $\mathrm{a}' \vdash \mathrm{c}$, and subalternation (or deducibility) $\mathrm{a} \vdash \mathrm{c}$.

4 Doing something different

These four relations play dissimilar roles above because our purpose was to generalize the (asymmetric) relation of sub-alternation; that is, to identify those functions that take the value 1 when c is deducible from a ($a \vdash c$). We could instead have begun with the (symmetric) relation of subcontrariety or independence, and sought to identify those functions $\mathfrak{f}(c \mid a) = \mathfrak{p}(c \mid a)$ that take the value 1 when $a' \vdash c$.

For this , the arduous investigation above would not need to be repeated (mutatis mutandis). Substitution of a' for a throughout the final table, followed by some straightforward Boolean transformations, yields everything that we want.

4 Eight measures of subcontrariety

The entries in the second and third columns are new, but the old function names in the first column undertake new duties, and now name the functions in the second column.

measure	in terms of $\mathfrak{p}$	when equal to $\mathbf{0}$
$\mathfrak{t}(c \mid a)$	$\mathfrak{p}(a \vee c \mid \top)$	$\nvdash a \vee c$ and $a \equiv c \equiv \bot$
$\mathfrak{b}(c \mid a)$	$\mathfrak{p}(a \vee c \mid a' \vee c')$	$\nvdash a \vee c$ and $a \equiv c$
$\mathfrak{a}(c \mid a)$	$\mathfrak{p}(c \mid a \to c)$	$\nvdash a \vee c$ and $c \equiv \bot$
$\mathfrak{s}(c \mid a)$	$\mathfrak{p}(a \mid c \to a)$	$\nvdash a \vee c$ and $a \equiv \bot$
$\mathfrak{c}(c \mid a)$	$\mathfrak{p}(c \mid a')$	$\nvdash a \vee c$ and $c \vdash a$
$\mathfrak{q}(c \mid a)$	$\mathfrak{p}(a \mid c')$	$\nvdash a \vee c$ and $a \vdash c$
$\mathfrak{n}(c \mid a)$	$\mathfrak{p}(a \mid a \triangle c)$	$\nvdash a \vee c$ and $a \vdash c'$
$\mathfrak{r}(c \mid a)$	$\mathfrak{p}(\bot \mid a'c')$	$\nvdash a \vee c$

5 *What happens when* $\mathfrak{p}$ *is updated*

It is trivial that when the probability measure $\mathfrak{p}$ is updated to $\mathfrak{p}_{\mathrm{b}}$ via the formula $\mathfrak{p}_{\mathrm{b}}(\mathrm{c} \,|\, \mathrm{a}) = \mathfrak{p}(\mathrm{c} \,|\, \mathrm{ab})$ (**Bayesian conditionalization**) then the credence measure $\mathfrak{c}$ is updated in the same way. In the case of contraprobability $\mathfrak{q}$, updating is achieved by the formula $\mathfrak{q}_{\mathrm{b}}(\mathrm{c} \,|\, \mathrm{a}) = \mathfrak{p}_{\mathrm{b}}(\mathrm{a}' \,|\, \mathrm{c}') = \mathfrak{p}(\mathrm{a}' \,|\, \mathrm{c}'\mathrm{b})$ $= \mathfrak{p}(\mathrm{b} \to \mathrm{c} \,|\, \mathrm{a})$. Our next task is to find a neat updating formula for the other six measures under consideration.

It will be shown, however, that no such formula exists for the measure $\mathfrak{t}(\mathrm{c} \,|\, \mathrm{a}) = \mathfrak{p}(\mathrm{a} \to \mathrm{c} \,|\, \top)$. This failure is an expression of the fact that, in general, the binary probability function is not definable in terms of the singulary function.

5 Formulas for updated measures

The table below presents the functions obtained from $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{s}$, $\mathfrak{c}$, $\mathfrak{q}$, $\mathfrak{n}$, and $\mathfrak{r}$ when the probability measure $\mathfrak{p}(c\,|\,a)$ from which they are defined is updated to $\mathfrak{p}_b(c\,|\,a) = \mathfrak{p}(c\,|\,ab)$.

measure	when updated by b
$\mathfrak{b}(c\,|\,a)$	$\mathfrak{b}_b(c\,|\,a) = \mathfrak{b}(b \to c\,|\,ba)$
$\mathfrak{a}(c\,|\,a)$	$\mathfrak{a}_b(c\,|\,a) = \mathfrak{a}(bc\,|\,ba)$
$\mathfrak{s}(c\,|\,a)$	$\mathfrak{s}_b(c\,|\,a) = \mathfrak{s}(b \to c\,|\,b \to a)$
$\mathfrak{c}(c\,|\,a)$	$\mathfrak{c}_b(c\,|\,a) = \mathfrak{c}(bc\,|\,ba) = \mathfrak{c}(b \to c\,|\,ba)$
$\mathfrak{q}(c\,|\,a)$	$\mathfrak{q}_b(c\,|\,a) = \mathfrak{q}(b \to c\,|\,ba) = \mathfrak{q}(b \to c\,|\,b \to a)$
$\mathfrak{n}(c\,|\,a)$	$\mathfrak{n}_b(c\,|\,a) = \mathfrak{n}(b \to c\,|\,b \to a) = \mathfrak{n}(bc\,|\,ba)$
$\mathfrak{r}(c\,|\,a)$	$\mathfrak{r}_b(c\,|\,a) = \mathfrak{r}(bc\,|\,ba) = \mathfrak{r}(b \to c\,|\,ba) = \mathfrak{r}(b \to c\,|\,b \to a)$

5 Alternative formulations

The table presents the updated measures in the most uniform manner possible, not necessarily the shortest. After all, we know already that $\mathfrak{c}_b(c\,|\,a)$ and $\mathfrak{q}_b(c\,|\,a)$ are most simply expressed as $\mathfrak{c}(c\,|\,ab)$ and as $\mathfrak{q}(b \rightarrow c\,|\,a)$. But it is striking that the four terms ba, bc, $b \rightarrow a$, and $b \rightarrow c$ are enough to express uniformly all the updated measures.

The proofs are in general rather cumbersome, though concerned only with Boolean equivalences. The correctness of the three expressions for $\mathfrak{r}_b(c\,|\,a)$ in terms of $\mathfrak{r}$ follow immediately, however, from the interdeducibility of the four terms $(ac')b$, $ba(bc)'$, $ba(b \rightarrow c)'$, and $(b \rightarrow a)(b \rightarrow c)'$.

5 Defining deducibility

The universally quantified formula $\forall b\, \mathfrak{p}(c\,|\,ba) = 1$ expresses correctly the deducibility of c from a. This relation is therefore expressed also by $\forall b\, \mathfrak{p}(a'\,|\,bc') = 1$, that is, by $\forall b\, \mathfrak{q}_b(c\,|\,a) = 1$, and accordingly by any one of the formulas $\forall b\, \mathfrak{q}(b \to c\,|\,a) = 1$, and $\forall b\, \mathfrak{q}(b \to c\,|\,ba) = 1$, and $\forall b\, \mathfrak{q}(b \to c\,|\,b \to a) = 1$. Similar definitions of deducibility in terms of $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{s}$, and $\mathfrak{n}$ are constructed in the same way.

The deducibility of c from a is definable similarly by equating to 1 for every b any of the expressions of $\mathfrak{r}_b(c\,|\,a)$ in terms of $\mathfrak{r}$, but it is palpable that the unquantified formulas $\mathfrak{r}(c \mid a) = 1$ and $\mathfrak{r}(c \mid a) \neq 0$ do the job equally well.

5 Padoa's method

Since every sentence is equivalent to a conditional, the task of defining deducibility using only the function $\mathfrak{t}$ reduces to that of defining deducibility in terms of absolute probability measures alone. An application of Padoa's method reveals that no such definition is possible. The two models give values for two measures μ(row, column) and ν(row, column).

μ	0	a	c	1
0	1	0	0	0
a	1	1	0	0
c	1	0	1	1
1	1	1	1	1

ν	0	a	c	1
0	1	1	0	0
a	1	1	0	0
c	1	1	1	1
1	1	1	1	1

5 Deducibility is not definable in terms of $\mathfrak{t}$

It is evident that although $\mu(\mathrm{b}\,|\,1) = \nu(\mathrm{b}\,|\,1)$ for each element b, and the four logically distinguishable elements in the first model are pairwise distinguishable in measure, the elements 0 and a in the second model are indistinguishable in measure, as are the elements c and 1; that is to say, $\nu(0\,|\,\mathrm{b}) = \nu(a\,|\,\mathrm{b})$ and $\nu(c\,|\,\mathrm{b}) = \nu(1\,|\,\mathrm{b})$ for all b.

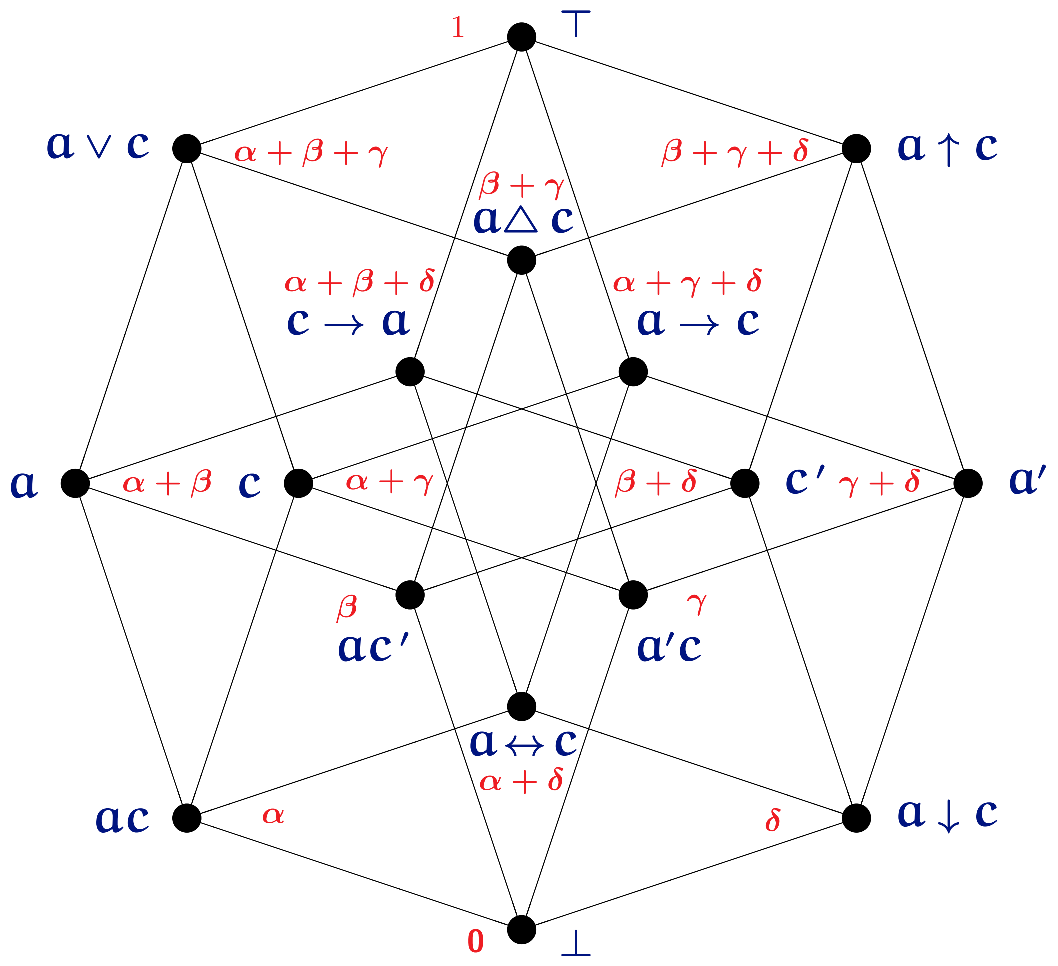
The two models agree on the values of $\mathfrak{p}(\mathrm{b}\,|\,\top)$, the absolute values of the probability measure $\mathfrak{p}$, but disagree on which elements are indistinguishable in measure, and accordingly interdeducible. This implies that deducibility has no definition in terms of absolute probability $\mathfrak{p}(\mathrm{b}\,|\,\top)$ alone.

6 Comparison of the eight functions

We have now shown that two of the eight functions $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{n}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$, namely the reductio function $\mathfrak{r}$ and the tautology function $\mathfrak{t}$, are not genuine quantitative generalizations of the relation of deducibility. $\mathfrak{r}(c \,|\, a)$ is two-valued, taking the value $\mathbf{0}$ when c is not deducible from a, and the value $\mathbf{1}$ when it is. And there is no function of $\mathfrak{t}(c \,|\, a)$ alone that determines whether or not c is deducible from a.

We shall, however, continue to regard $\mathfrak{r}$ and $\mathfrak{t}$ as (degenerate) measures of degree of deducibility. It will be shown that the eight functions, ordered by magnitude, arrange themselves elegantly into an 8-element Boolean lattice (a cube).

6 The free Boolean algebra on a and c



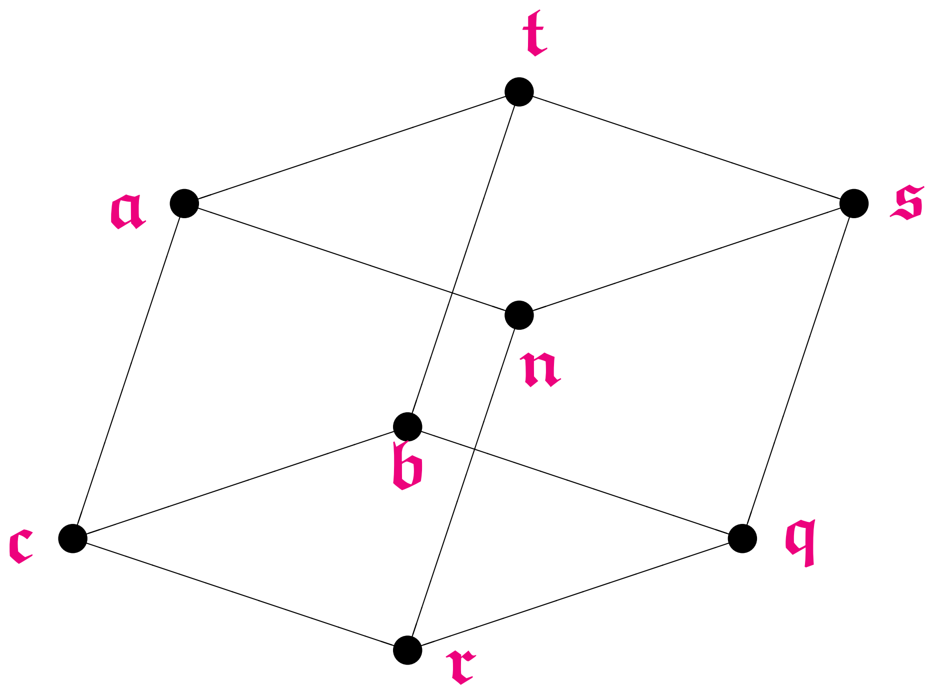
6 Numerical comparisons

Using this drawing of the tesseract, we can easily compute numerical values for the eight measures in the finite case in which α, β, γ, δ are positive (and $\alpha + \beta + \gamma + \delta = 1$).

$\mathfrak{t}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{a} \to \mathsf{c} \mid \top)$	$=$	$\alpha + \gamma + \delta$
$\mathfrak{b}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{a} \leftrightarrow \mathsf{c} \mid \mathsf{c} \to \mathsf{a})$	$=$	$(\alpha + \delta)/(\alpha + \beta + \delta)$
$\mathfrak{a}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{c} \mid \mathsf{a} \vee \mathsf{c})$	$=$	$(\alpha + \gamma)/(\alpha + \beta + \gamma)$
$\mathfrak{s}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{a}' \mid \mathsf{a} \uparrow \mathsf{c})$	$=$	$(\gamma + \delta)/(\beta + \gamma + \delta)$
$\mathfrak{c}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{ac} \mid \mathsf{a})$	$=$	$\alpha/(\alpha + \beta)$
$\mathfrak{q}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{a} \downarrow \mathsf{c} \mid \mathsf{c}')$	$=$	$\delta/(\beta + \delta)$
$\mathfrak{n}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\mathsf{a}'\mathsf{c} \mid \mathsf{a} \triangle \mathsf{c})$	$=$	$\gamma/(\beta + \gamma)$
$\mathfrak{r}(\mathsf{c} \mid \mathsf{a})$	$\mathfrak{p}(\bot \mid \mathsf{ac}')$	$=$	$\mathbf{0}$

6 Ordering by magnitude

In the figure below the function $\mathfrak{f}$ is located below the function $\mathfrak{h}$ if & only if $\mathfrak{f}(\mathfrak{c}\,|\,\mathfrak{a}) \leq \mathfrak{h}(\mathfrak{c}\,|\,\mathfrak{a})$ for all values of α, β, γ, δ, and also $\mathfrak{f}(\mathfrak{c}\,|\,\mathfrak{a}) < \mathfrak{h}(\mathfrak{c}\,|\,\mathfrak{a})$ for some α, β, γ, δ.



6 The infinite case

What happens in a much larger algebra in which some, or all, of α, β, γ, δ have the value $\mathbf{0}$? Here we may take advantage of the well known expansion of the range of the probability function $\mathfrak{p}$ from the real numbers to a non-archimedean extension field containing infinitesimals. Compactness ensures that provided that the extension field is large enough, every Boolean algebra supports a regular probability measure $\mathfrak{p}$.

What we need here is rather more: the theorem (if it is a theorem) that every measure $\mathfrak{p}$ satisfying Popper's axioms is approximated pointwise (within an infinitesimal) by a regular non-archimedean measure that also satisfies the axioms.

6 A less dirty proof

The lattice ordering of the eight functions $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{n}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$ is thus assured. You will be pleased to know, however, that a more elementary proof of the ordering is possible.

It was known to Reichenbach (1949) and also to Popper that $\mathfrak{c}(c\,|\,a) \leq \mathfrak{t}(c\,|\,a)$; that is, that the so-called conditional probability is never greater than the probability of the material conditional. That $\mathfrak{r}(c\,|\,a) \leq \mathfrak{c}(c\,|\,a)$ follows from the two-valuedness of $\mathfrak{r}$. In (1987) Popper & Miller showed that $\mathfrak{a}(c\,|\,a)$ can be interpolated between $\mathfrak{c}(c\,|\,a)$ and $\mathfrak{t}(c\,|\,a)$, whence $\mathfrak{r}(c\,|\,a) \leq \mathfrak{c}(c\,|\,a) \leq \mathfrak{a}(c\,|\,a) \leq \mathfrak{t}(c\,|\,a)$. There are similar proofs for the other chains from $\mathfrak{r}(c\,|\,a)$ to $\mathfrak{t}(c\,|\,a)$.

7 The functions $\mathfrak{c}$, $\mathfrak{q}$, and $\mathfrak{n}$,

It has been shown elsewhere that the functions $\mathfrak{c}$ (credence) and $\mathfrak{q}$ (contraprobability) have sometimes been confused with one another. Indeed, I suggest that $\mathfrak{q}$, which is a measure of deductive dependence or approximation is almost always misidentified as a measure of probability, especially in courts of law. They are actually quite easy to mix up.

It is natural to suppose therefore that $\mathfrak{c}$ and $\mathfrak{q}$ must also have something in common with $\mathfrak{n}$, the third measure adjacent to $\mathfrak{r}$. But what do they have in common? Recall that $\mathfrak{n}(c\,|\,a)$ takes the value 1 when c is deducible from a, and the value 0 when it is not deducible from a but a is deducible from c.

7 The nearness (or proximity) function $\mathfrak{n}$

Formally $\mathfrak{n}$ is a **proximity** function; that is, its complement $\mathfrak{d}(c\,|\,a) = 1 - \mathfrak{n}(c\,|\,a)$ is a **directed pseudometric**. The non-standard method just mentioned enables a general proof of the directed **triangle inequality** $\mathfrak{d}(c\,|\,b) + \mathfrak{d}(b\,|\,a) \geq \mathfrak{d}(c\,|\,a)$. Indeed, since $\mathfrak{d}(c\,|\,a) + \mathfrak{d}(a\,|\,c) = 1$, it is only $\mathfrak{n}(b\,|\,b) = 1$ that prevents $\mathfrak{n}(a\,|\,c)$ from also being a pseudometric.

In a passage in his *Realism* Popper managed to conflate $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{n}$ when he wrote: '$\mathfrak{p}(c \mid a) = r$ is an assertion about the **contents** of c and a and their **degree** of **logical proximity**; or more precisely, about the degree to which the statement c contains information which is contained in a.'

7 Three respects in which $\mathfrak{n}$ differs from $\mathfrak{c}$ and $\mathfrak{q}$

Popper provided an autonomous axiom system for $\mathfrak{c}$, and using the identity $\mathfrak{q}(c\,|\,a) = \mathfrak{p}(a'\,|\,c')$ we easily write down an autonomous system for $\mathfrak{q}$. Nothing like this is known for $\mathfrak{n}$.

The Dutch book argument in favour of the function $\mathfrak{c}$ appeals to betting rates. An economic argument in favour of $\mathfrak{q}$ appeals to discount rates. Nothing like this is known for $\mathfrak{n}$.

$\mathfrak{c}(c\,|\,a) = 0$ says that none of the semantic content (countermodels) of c is contained in that of a. $\mathfrak{q}(c\,|\,a) = 0$ says that none of the syntactic content (consequences) of c is contained in that of a. Nothing like this is known for $\mathfrak{n}$.

7 Linguistic transformation

That the measures $\mathfrak{c}$, $\mathfrak{q}$, and $\mathfrak{n}$ are on a par is suggested by the fact that by moving from the language based on the letters a and c to another interdeducible language we can convert each of $\mathfrak{c}$, $\mathfrak{q}$, and $\mathfrak{n}$ to one of the others.

Let b and d be interdeducible respectively with c' and $a \leftrightarrow c$. Then c is interdeducible with b', and a, which is interdeducible with $(a \leftrightarrow c) \leftrightarrow c$, is interdeducible also with $b \triangle d$. Substituting in the formulas for $\mathfrak{c}$, $\mathfrak{q}$, and $\mathfrak{n}$, and simplifying, we find that $\mathfrak{c}(c\,|\,a) = \mathfrak{n}(d\,|\,b)$, $\mathfrak{q}(c\,|\,a) = \mathfrak{c}(d\,|\,b)$, and $\mathfrak{n}(c\,|\,a) = \mathfrak{q}(d\,|\,b)$. The final slide shows that other equivalent languages allow $\mathfrak{c}$, $\mathfrak{q}$, and $\mathfrak{n}$ to be shuffled at will.

7 A table of transformations

To transform	$\mathfrak{c}$ $\mathfrak{q}$ $\mathfrak{n}$	
into	$\mathfrak{c}$ $\mathfrak{q}$ $\mathfrak{n}$	set $b \equiv a$ and $d \equiv c$
	$\mathfrak{c}$ $\mathfrak{n}$ $\mathfrak{q}$	$b \equiv a$ and $d \equiv a \leftrightarrow c$
	$\mathfrak{q}$ $\mathfrak{c}$ $\mathfrak{n}$	$b \equiv c'$ and $d \equiv a'$
	$\mathfrak{q}$ $\mathfrak{n}$ $\mathfrak{c}$	$b \equiv a \triangle c$ and $d \equiv a'$
	$\mathfrak{n}$ $\mathfrak{c}$ $\mathfrak{q}$	$b \equiv c'$ and $d \equiv a \leftrightarrow c$
	$\mathfrak{n}$ $\mathfrak{q}$ $\mathfrak{c}$	$b \equiv a \triangle c$ and $d \equiv c$

The values of any of the measures $\mathfrak{c}$, $\mathfrak{q}$, $\mathfrak{n}$ become the values, for different arguments, of one of the other measures.