Universal-existential sentences in the simple theory of types

joint work with Anuj Dawar and Thomas Forster

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- What about sentences with four quantifiers?

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Well-formed formulae of $\mathcal{L}_{\mathrm{TST}}$ *are built up inductively from the atomic formulae* $x_n^i \in_i y_m^{i+1}$ *and* $x_n^i =_i y_m^i$ *etc.*

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An $\mathcal{L}_{\mathrm{TST}}$ structure $\mathcal{M}$ consists of a sequence of domains $\langle M_i \mid i \in \mathbb{N} \rangle$ and a sequence $\langle \in_i \mid i \in \mathbb{N} \rangle$ such that $\in_i \subseteq M_i \times M_{i+1}$ for all $i \in \mathbb{N}$. We will normally interpret the relations $=_i$ as true equality in the domains M_i. Quantifiers range over the domain M_i where i indicated by the superscript of the variable to the right of the quantifier.

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(Comprehension) for all $i \in \mathbb{N}$ *and for all* $\mathcal{L}_{\mathrm{TST}}$*-formulae* $\phi(x^i, \vec{z})$*,*

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- If $\mathcal{M}$ is an $\mathcal{L}_{\mathrm{TST}}$ structure with $|M_0| = n$ then we say that $\mathcal{M}$ is finitely generated by n atoms

The class of universal-existential and existential-universal sentences

Definition

We use $\forall^*\exists^*$ *to denote the class of well-formed* $\mathcal{L}_{\mathrm{TST}}$ *sentences in the form*

$$\forall x_1^{r_1} \ldots \forall x_k^{r_k} \exists y_1^{s_1} \ldots \exists y_l^{s_l} \theta$$

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Conjecture

The theory TST *decides every sentence in the class* $\forall^*\exists^* \cup \exists^*\forall^*$*.*

$\exists^*\forall^*$ sentences and finitely generated models

Theorem

Let $\phi = \exists x_1^{r_1} \ldots \exists x_k^{r_k} \forall y_1^{s_1} \ldots \forall y_l^{s_l} \theta$ be an $\mathcal{L}_{\mathrm{TST}}$ sentence with θ quantifier-free and $r_1 \leq \ldots \leq r_k$. If $\mathcal{N} \models \mathrm{TST} + \phi$ and $\mathcal{M} = \langle M_0, M_1, \ldots \rangle$ is finitely generated with $\mathcal{M} \models \mathrm{TST}^-$ and $|M_0| \geq \mathbf{G}_k(r_k)$ then $\mathcal{M} \models \phi$.

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- Define $\mathbf{G}_k : \mathbb{N} \longrightarrow \mathbb{N}$ by $\mathbf{G}_k(0) = k$ and $\mathbf{G}_k(n+1) = \mathbf{G}_k(n) + \binom{\mathbf{G}_k(n)}{2}$.

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Lemma

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(i) *$f_n : N_n \longrightarrow M_n$ is an injection,*

(ii) *for all $x \in N_n$ and for all $y \in N_{n+1}$,*

$$\mathcal{N} \models x \in y \textit{ if and only if } \mathcal{M} \models f_n(x) \in f_{n+1}(y),$$

(iii)

$$a_1^{r_1}, \ldots, a_k^{r_k} \in \bigcup_{n \in \mathbb{N}} \mathrm{rng}(f_n).$$

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- Our result shows that no $\exists^*\forall^*$ sentence is an “axiom of infinity”
- If there is no sentence in $\forall^*\exists^*$ which is an “axiom of infinity” then TST decides every sentence in $\exists^*\forall^* \cup \forall^*\exists^*$

The decidability of a subclass of $\forall^*\exists^*$

- Let $\phi = \forall x_1^{r_1} \ldots \forall x_k^{r_k} \exists y_1^{s_1} \ldots \exists y_l^{s_l} \theta$ with $r_1 < \ldots < r_k$, $s_1 > \ldots > s_l$ and θ quantifier-free

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  - for all $\alpha \in \mathcal{C}_{i-1,j}$, the existence of an element $y \in x$ with $c_{i-1,j}^{\mathcal{M}}(y) = \alpha$ and the existence of an element $y \notin x$ with $c_{i-1,j}^{\mathcal{M}}(y) = \alpha$

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Some definitions

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Let $m \in \mathbb{N}$. We say that a colour $\alpha \in \mathcal{C}_{i,j}$ is m-special with respect to a colouring $f : X \longrightarrow \mathcal{C}_{i,j}$ if and only if $|\{x \in X \mid f(x) = \alpha\}| = m$. If $\alpha \in \mathcal{C}_{i,j}$ is 0-special then we say that α is forbidden.

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Let $m \in \mathbb{N}$. We say that a colour $\alpha \in \mathcal{C}_{i,j}$ is m-special with respect to a colouring $f : X \longrightarrow \mathcal{C}_{i,j}$ if and only if $|\{x \in X \mid f(x) = \alpha\}| = m$. If $\alpha \in \mathcal{C}_{i,j}$ is 0-special then we say that α is forbidden.

Definition

Let $m \in \mathbb{N}$. We say that a colour $\alpha \in \mathcal{C}_{i,j}$ is m-abundant with respect to a colouring $f : X \longrightarrow \mathcal{C}_{i,j}$ if and only if $|\{x \in X \mid f(x) = \alpha\}| \geq m$.

Definition

Let $K \in \mathbb{N}$. We say that colourings $f : X \longrightarrow \mathcal{C}_{i,j}$ and $g : Y \longrightarrow \mathcal{C}_{i,j}$ are K-similar if and only if for all $0 \leq m < K$ and for all $\alpha \in \mathcal{C}_{i,j}$,

$$\alpha \textit{ is } m\textit{-special w.r.t. } f \textit{ if and only if } \alpha \textit{ is } m\textit{-special w.r.t. } g.$$

The base case

- We will pick $b_1^{r_1}, \ldots, b_k^{r_k} \in \mathcal{N}$ so as to ensure that $c_{i,k}^{\mathcal{M}}$ and $c_{i,k}^{\mathcal{N}}$ are 2^2-similar for all $i \in \mathbb{N}$

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- Let $\mathcal{C}_{0,0} = \{0\}$ and define $c_{0,0}^{\mathcal{M}} : M_0 \longrightarrow \mathcal{C}_{0,0}$ and $c_{0,0}^{\mathcal{N}} : N_0 \longrightarrow \mathcal{C}_{0,0}$ by

$$c_{0,0}^{\mathcal{M}}(x) = 0 \text{ for all } x \in M_0$$

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The base case

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- Note that $c_{0,0}^{\mathcal{M}}$ and $c_{0,0}^{\mathcal{N}}$ are 2^{k+2}-similar

Recursion I

- Suppose that $\mathcal{C}_{n,0}$, $c^{\mathcal{M}}_{n,0} : M_n \longrightarrow \mathcal{C}_{n,0}$ and $c^{\mathcal{N}}_{n,0} : N_n \longrightarrow \mathcal{C}_{n,0}$ have been defined and $c^{\mathcal{M}}_{n,0}$ and $c^{\mathcal{N}}_{n,0}$ are 2^{k+2}-similar

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- Suppose $\mathcal{C}_{n,0} = \{\alpha_1, \ldots, \alpha_q\}$
- Define $\mathcal{C}_{n+1,0} = 2^{2 \cdot q}$

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- Define $\mathcal{C}_{n+1,0} = 2^{2\cdot q}$
- Define $c^{\mathcal{M}}_{n+1,0} : M_{n+1} \longrightarrow \mathcal{C}_{n+1,0}$ such that for all $x \in M_{n+1}$,

$$c^{\mathcal{M}}_{n+1,0}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$$

$$\text{where } f_i = \begin{cases} 0 & \text{if for all } y \in M_n, \text{ if } c^{\mathcal{M}}_{n,0}(y) = \alpha_i \text{ then } y \notin x \\ 1 & \text{if there is } y \in M_n, \text{ s.t. } c^{\mathcal{M}}_{n,0}(y) = \alpha_i \text{ and } y \in x \end{cases}$$

$$\text{and } g_i = \begin{cases} 0 & \text{if for all } y \in M_n, \text{ if } c^{\mathcal{M}}_{n,0}(y) = \alpha_i \text{ then } y \in x \\ 1 & \text{if there is } y \in M_n, \text{ s.t. } c^{\mathcal{M}}_{n,0}(y) = \alpha_i \text{ and } y \notin x \end{cases}$$

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- Define $c^{\mathcal{N}}_{n+1,0} : N_{n+1} \longrightarrow \mathcal{C}_{n+1,0}$ identically

Recursion II

- Suppose that for all $n \in \mathbb{N}$, $c^{\mathcal{M}}_{n,j}$, $c^{\mathcal{N}}_{n,j}$ and $\mathcal{C}_{n,j}$ have been defined, and $c^{\mathcal{M}}_{n,j}$ and $c^{\mathcal{N}}_{n,j}$ are 2^{k-j+2}-similar

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- Suppose that for all $n \in \mathbb{N}$, $c_{n,j}^{\mathcal{M}}$, $c_{n,j}^{\mathcal{N}}$ and $\mathcal{C}_{n,j}$ have been defined, and $c_{n,j}^{\mathcal{M}}$ and $c_{n,j}^{\mathcal{N}}$ are 2^{k-j+2}-similar
- Suppose that $b_1^{r_1}, \ldots, b_j^{r_j} \in \mathcal{N}$ are the elements used to define $c_{n,j}^{\mathcal{N}}$

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- Suppose that $b_1^{r_1}, \ldots, b_j^{r_j} \in \mathcal{N}$ are the elements used to define $c^{\mathcal{N}}_{n,j}$
- For each $n < r_{j+1} - 1$, define $\mathcal{C}_{n,j+1} = \mathcal{C}_{n,j}$, $c^{\mathcal{M}}_{n,j+1} = c^{\mathcal{M}}_{n,j}$ and $c^{\mathcal{N}}_{n,j+1} = c^{\mathcal{N}}_{n,j}$

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- Define $\mathcal{C}_{r_{j+1}-1,j+1} = 2 \times \mathcal{C}_{r_{j+1}-1,j}$

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- Define $\mathcal{C}_{r_{j+1}-1,j+1} = 2 \times \mathcal{C}_{r_{j+1}-1,j}$
- Suppose $\mathcal{C}_{r_{j+1}-2,j} = \{\alpha_1, \ldots, \alpha_q\}$

Recursion II

- Define $c^{\mathcal{M}}_{r_{j+1}-1,j+1} : M_{r_{j+1}-1} \longrightarrow \mathcal{C}_{r_{j+1}-1,j+1}$ such that for all $x \in M_{r_{j+1}-1}$,

$$c^{\mathcal{M}}_{r_{j+1}-1,j+1}(x) = \langle F, f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$$

$$\text{where } c^{\mathcal{M}}_{r_{j+1}-1,j}(x) = \langle f_1, \ldots, f_q, g_1, \ldots, g_q \rangle$$

$$\text{and } F = \begin{cases} 1 & \text{if } x \in a^{r_{j+1}}_{j+1} \\ 0 & \text{if } x \notin a^{r_{j+1}}_{j+1} \end{cases}$$

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- For $n \geq r_{j+1}$ can define $\mathcal{C}_{n,j+1}$, $c^{\mathcal{M}}_{n,j+1}$ and $c^{\mathcal{N}}_{n,j+1}$ the same way we did in the base case

A decidable subclass of $\forall^*\exists^*$

Theorem

$\mathcal{M} \models \phi$

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Proof.
Since $\mathcal{N} \models \phi$, we can find $c_1^{s_1}, \ldots, c_l^{s_l} \in \mathcal{N}$ such that $\mathcal{N} \models \theta(b_1^{r_1}, \ldots, b_k^{r_k}, c_1^{s_1}, \ldots, c_l^{s_l})$.

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Inductively choose $d_1^{s_1}, \ldots, d_l^{s_l} \in \mathcal{M}$ such that $\mathcal{M} \models \theta(a_1^{r_1}, \ldots, a_k^{r_k}, d_1^{s_1}, \ldots, d_l^{s_l})$. □

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Theorem

If $\phi = \forall x_1^{r_1} \ldots \forall x_k^{r_k} \exists y_1^{s_1} \ldots \exists y_l^{s_l} \theta$ is an $\mathcal{L}_{\mathrm{TST}}$ sentence with $r_1 < \ldots < r_k$, $s_1 > \ldots > s_l$ and θ quantifier-free then TST decides ϕ.

Extending this argument

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