Truth and Collection in nonstandard models of PA

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Introduction

- Let $M \models \mathrm{PA}$ be nonstandard.
- Tarski's definition gives a notion of truth $T \subseteq M$ such that

$$M \models T(\ulcorner \sigma \urcorner) \Leftrightarrow M \models \sigma$$

  for standard $\sigma \in \mathscr{L}_A$
- We also have Tarski conditions:

$$M \models T(\ulcorner \forall x\ \sigma \urcorner) \leftrightarrow \forall x\ T(\ulcorner \sigma(\mathrm{clterm}\, x) \urcorner)$$

  etc.
- What can we say about the truth of nonstandard $\sigma \in M$?

Full satisfaction classes

(Note: this subject is treated in the literature as notions of *satisfaction* with lists of values for free variables. We continue to treat in the equivalent way as a notion of truth for sentences, with substitition and canonical terms.)

- $T \subseteq M$ is a *full satisfaction class* if it defines truth for all sentences of $\mathscr{L}_A$ in the sense of M, satisfies the Tarski conditions for each of the connectives and agrees with the usual definition of truth for atomic sentences.
- M will be taken throughout to be nonstandard and countable.
- A theorem of Lachlan says that not all such M have such a full satisfaction class.

for nonstandard M, M has a full satisfaction class implies M is recursively saturated

Recursive saturation

- M is *recursively saturated* if it satisfies all recursive sentences of the form

$$\forall \bar{a} \left(\bigwedge_n \exists x \bigwedge\nolimits_{i<n} \theta_i(x,\bar{a}) \to \exists x \bigwedge\nolimits_i \theta_i(x,\bar{a}) \right).$$

- M is *short recursively saturated* if it satisfies

$$\forall \bar{a}, b \left(\bigwedge_n \exists x < b \bigwedge\nolimits_{i<n} \theta_i(x,\bar{a}) \to \exists x < b \bigwedge\nolimits_i \theta_i(x,\bar{a}) \right).$$

- M is *tall* if it satisfies

$$\forall \bar{a} \left(\bigwedge_n \exists y\ \theta_n(y,\bar{a}) \to \exists b \bigwedge\nolimits_n \exists y < b\ \theta_n(y,\bar{a}) \right).$$

Recursive saturation

- M recursively saturated implies M is short recursively saturated and tall.
- Conversely, if M is short recursively saturated and tall then it is recursively saturated.
- Examples where M is recursively saturated, tall but not short recursively saturated, short recursively saturated but not tall, or neither, are all easy to find.

Kotlarski–Krajewski–Lachlan (KKL) Theorem

- Any countable recursively saturated M has a full satisfaction class.
- In fact it has continuum many such classes.
- The set of statements true in all such classes is defined by provability in M-logic:

$$\sigma \in T \text{ for all full satisfaction classes } T \quad \Leftrightarrow \quad \vdash_M \sigma$$

M-logic

- M-logic has the cut rule:

  $$\text{from } \vdash_M \Delta, \sigma \text{ and } \vdash_M \Delta, \neg\sigma \text{ deduce } \vdash_M \Delta$$

  so that if Γ is finite and M-consistent then so is either Γ, σ or $\Gamma, \neg\sigma$.
- M-logic also has the infinitary rule:

  $$\text{if } \vdash_M \Delta, \theta(\mathrm{clterm}\, a) \text{ for all } a \in M \text{ then } \vdash_M \Delta, \forall x\, \theta(x)$$

  which is the main source of difficulty.

Consistency of M-logic

- By countability of M it is easy to show that if Γ is finite and M-consistent then there is $T \supseteq \Gamma$ which is a full satisfaction class.
- The consistency of M-logic is proved in the KKL Theorem in the case when M is recursively saturated by reducing it to a finitary logic $\vdash_{FA}$. The hard step is to use saturation to simulate the M-rule.
- Fact (Smith): M-logic is consistent if and only if M is recursively saturated.

The problem

- The reduction of $\vdash_M$ to $\vdash_{FA}$ shows that the set S_Γ of all satisfaction classes T extending a given finite M-consistent Γ is finitely based. By topologising, the complement of S_Γ is nowhere dense, and the set of all such T is comeagre (i.e. large in the sense of Baire).
- For any M-consistent Γ and any σ it is 'easy to see' any sufficiently long

$$(\sigma \vee (\sigma \vee (\sigma \vee (\sigma \vee \cdots))))$$

  is consistent with Γ. This includes the case when σ is 'obviously' false, such as $0 = 1$.
- Thus, when they exist, almost all T (in the sense of Baire) contain 'pathological' incorrect statements.

Stronger logics

- We search for stronger conditions that eliminate these pathological examples.
- One well-known possibility is add the induction scheme in $\mathscr{L}_T$ to the axioms PA plus Tarski's conditions on T.
- However this results in additional $\mathscr{L}_A$-consequences. In particular $\mathrm{Con}(\mathrm{PA})$ is provable in such a system.
- $\mathrm{PA}(S)$ and $I\Delta_0(S)$ are two natural systems that have been investigated. They eliminate pathologies, but are too strong.

Desiderata

- It would be particularly interesting to find additional conditions on truth that have no additional $\mathscr{L}_A$-consequences that work with the Tarski conditions to eliminate pathological examples.
- Unless the additional conditions are nonrecursive, the same connection with recursive saturation is expected (because of a well-known phenomenon called resplendency).

A suggestion, and dead end

- Since all the pathological examples are known to be equivalent (in propositional logic) to statements that are false, a natural suggestion is to ask that T be closed under propositional logic in the sense of M.
- Unfortunately, this has been shown to be equivalent to $I\Delta_0(S)$ and hence $\mathrm{Con}(\mathrm{PA})$ is provable in such a system. (Idea: simulate bounded quantifiers by propositional logic.)

$I\Delta_0(S)$

The theory $I\Delta_0(S)$ arises in a surprising number of ways.

- PA + Tarski conditions + induction on all Δ_0 formulas of $\mathscr{L}_T$.
- (Kotlarski) PA + Tarski conditions + T contains all theorems of PA.
- (Cieśliński) PA + Tarski conditions + T is closed under proofs in propositional logic.

The theory is not conservative over PA.

Elimination of some pathologies

As observed by Engström,

- PA + Tarski conditions + the scheme that 'T agrees with Tr_{Σ_n} for each standard n' is conservative over PA.

However this does not eliminate all 'obvious' pathologies, since some quantifier prefixes may be easy to read but not of bounded complexity.

Collection

- The collection principle is

$$\forall x < t \; \exists y \; \theta(x, y) \to \exists z \; \forall x < t \; \exists y < z \; \theta(x, y)$$

- With a small amount of induction it is equivalent to full induction: $I\Delta_0 + \mathrm{Coll} = \mathrm{PA}$.
- However without induction it is weak: $\mathrm{PA}^- + \mathrm{Coll} \equiv_{\Pi_1} \mathrm{PA}^-$.
- Can we ensure all collection axioms are made true by T?
- What is the strength of $\mathrm{PA} + \mathrm{Tarski} + \mathrm{Coll}(\mathscr{L}_T)$?

Making nonstandard collection true

- 90% Theorem[1]: if M is countable recursively saturated there is a full satisfaction class T on M containing all nonstandard $\mathscr{L}_A$ collection axioms in the sense of M.
- Problem: this is proved using the same notion of consistency. So all the same pathologies can still be added consistently at any stage of the construction.
- (Kotlarski observed that, in a similar way, all nonstandard induction axioms can be made true.)

[1] Detailed checking required.

Making nonstandard collection true: the idea

- Look at the finite approximation $\vdash_{FA}$ to $\vdash_M$. This is of the form $\bigvee_{n\in\mathbb{N}} \theta_n$ (recursive).
- We want to simulate the rule 'from $\forall x < t\ \exists y\ \phi(x,y)$ deduce $\exists z\ \forall x < t\ \exists y < z\ \phi(x,y)$'.
- By substitutions and syntactic operations (to be checked) this becomes

$$\text{`}\vdash_{FA} \Delta, \forall x < t\ \exists y\ \phi(x,y)\ \textit{implies}$$
$$\vdash_{FA} \Delta, \exists z\ \forall x < t\ \exists y < z\ \phi(x,y)\text{'}$$

or

$$\text{`}\forall x < t\ \exists y\ (\vdash_{FA} \Delta, \phi(\mathrm{clterm}\, x, \mathrm{clterm}\, y))\ \textit{implies}$$
$$\exists z\ \forall x < t\ \exists y < z\ (\vdash_{FA} \Delta, \phi(\mathrm{clterm}\, x, \mathrm{clterm}\, y))\text{'}$$

Infinitary collection

For recursive conjunctions and disjunctions of $\mathscr{L}_A$ formulas $\mathrm{Coll}\bigvee$ and $\mathrm{Coll}\bigwedge$ are defined as

- $\mathrm{Coll}\bigvee$: $\forall x < t\ \exists y\ \bigvee_i \theta_i(x, y)$ implies $\exists z\ \forall x < t\ \exists y < z\ \bigvee_i \theta_i(x, y)$.
- $\mathrm{Coll}\bigwedge$: $\forall x < t\ \exists y\ \bigwedge_i \theta_i(x, y)$ implies $\exists z\ \forall x < t\ \exists y < z\ \bigwedge_i \theta_i(x, y)$.
- Both are consequences of recursive saturation.
- (The similar ideas of induction applied to $\bigvee$ or $\bigwedge$ are not true except for the standard model.)

$\mathrm{Coll}\bigvee$ and $\mathrm{Coll}\bigwedge$

- $\mathrm{Coll}\bigwedge$ implies Tall and seems to (at least very nearly) be the same as Tall. (Question: is this an equivalence?). Non short-rec-saturated models of $\mathrm{Coll}\bigwedge$ are easy to build.
- $\mathrm{Coll}\bigvee$ is implied by Tall and also by Short Rec Sat. In fact

$$\mathrm{Coll}\bigvee \Leftrightarrow (\mathsf{Tall} \vee \mathsf{Short\ Rec\ Sat}).$$

- $\mathrm{Coll}\bigvee$ is what we use.

Making collection in the extended language true

- Conjecture: if M is countable recursively saturated there is a full satisfaction class T on M such that $(M, T) \models \mathrm{Coll}(\mathscr{L}_T)$.
- 0% Idea: extend M to a countable recursively saturated ω_1-like $M_1 \succ_{\mathrm{e}} M$. Apply the KKL construction to M_1, iterating through ordinals $\alpha < \omega_1$. Then there is T_1 such that $(M_1, T_1) \models \mathrm{Coll}(\mathscr{L}_T)$.
- Fact (Smith): there are ω_1-like rec sat models with no full satisfaction classes.

Questions

- What (if anything:) do these collection axioms actually do? (They are closely related to the idea of cofinal sets and largeness properties. And through this to the idea of end extensions of models. Does this help?)
- What is the status of $\mathrm{Coll}(\mathscr{L}_T)$, i.e. the collection scheme expressed in the language $\mathscr{L}_T$?
- Can one say anything about the collection schemes $\mathrm{Coll}\bigvee\bigwedge$ and $\mathrm{Coll}\bigwedge\bigvee$?
- What arithmetical schemes other than Coll have plausible $\mathscr{L}^{\mathrm{rec}}_{\omega_1\omega}$ versions?